Spontaneous $\mathcal{PT}$ symmetry breaking and pseudo-supersymmetry

2006 J. Phys. A: Math. Gen. 39 L377

(http://iopscience.iop.org/0305-4470/39/23/L01)

J. Phys. A: Math. Gen. **39** (2006) L377–L384 doi:10.1088/0305-4470/39/23/L01

LETTER TO THE EDITOR

Spontaneous $\mathcal{PT}$ symmetry breaking and pseudo-supersymmetry

A Sinha and P Roy

Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata-700 108, India

E-mail: anjana_t@isical.ac.in and pinaki@isical.ac.in

Received 14 February 2006
Published 23 May 2006
Online at stacks.iop.org/JPhysA/39/L377

Abstract
The phenomena of spontaneous $\mathcal{PT}$ symmetry breaking, associated with non-Hermitian Hamiltonians, are investigated. It is shown that spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of *pseudo-supersymmetry*. We also discuss in detail the resulting structure.

PACS number: 03.65.−w

1. Introduction

Non-Hermitian quantum mechanics has drawn a lot of attention for almost a decade now, because of the intrinsic interest of such potentials [1] admitting real spectrum under certain conditions, as well as their possible applications [2–4]. Among the various non-Hermitian models, a particular class with $\mathcal{PT}$ symmetry is of special interest, since their energy spectrum exhibits a characteristic feature—the energies are real for unbroken $\mathcal{PT}$ symmetry (when the potential as well as the wavefunctions are invariant under the combined action of space inversion ($\mathcal{P}$) and time reversal ($\mathcal{T}$)) while they switch to complex conjugate pairs for spontaneously broken $\mathcal{PT}$ symmetry (i.e., the wavefunctions lose their $\mathcal{PT}$ symmetry, although the potential still retains it) [5–7]. At the same time, various studies have shown that $\mathcal{PT}$ symmetry is neither a necessary nor a sufficient condition for the existence of a real spectrum. The criteria for the energies to be real (or in complex conjugate pairs) are the η-pseudo-Hermiticity of these non-Hermitian Hamiltonians [8].

The phenomenon of spectral discontinuity has been the subject of study of a number of works, both for Hermitian models [9, 10] as well as non-Hermitian ones [6, 8, 11–13], employing a variety of techniques. In particular, it has been observed that it occurs when a set of parameters in the potential reaches certain critical values. While the nonanalytic behaviour of the energy spectrum was interpreted in terms of supersymmetry breaking in Hermitian systems [10], an interplay was established between $\mathcal{PT}$ symmetry and supersymmetry in a certain class of non-Hermitian models [12–14]. In the present letter, we shall show that the spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of *pseudo-supersymmetry*, and establish the significant role played by a set of parameters a (in the non-Hermitian potential) in this respect. We shall make a detailed study with the help of a couple of exactly solvable examples, and also study the nature of the wavefunctions.

2. Theory

To begin with let us briefly recall some bare facts about $\mathcal{PT}$ symmetry. A non-Hermitian Hamiltonian $H(x;a)$, given by (a denoting a set of parameters)

$$H(x;a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x;a) \tag{1}$$

is said to be $\mathcal{PT}$ symmetric if

$$(\mathcal{PT})H(x;a) = H(x;a)(\mathcal{PT}) \tag{2}$$

where the *space inversion* operator $\mathcal{P}$ and the *time reversal* operator $\mathcal{T}$ are defined by their action on the position, momentum and identity operators, respectively, as

$$\mathcal{P}x\mathcal{P} = -x, \qquad \mathcal{P}p\mathcal{P} = \mathcal{T}p\mathcal{T} = -p, \qquad \mathcal{T}(i.1)\mathcal{T} = -i.1 \tag{3}$$

We note that for unbroken $\mathcal{PT}$ symmetry, the Hamiltonian $H(x;a)$ and the wavefunctions $\psi(x;a)$ are both invariant under the $\mathcal{PT}$ transformations [6, 7]

$$H^*(-x;a) = H(x;a), \qquad \psi^*(-x;a) = \pm\psi(x;a). \tag{4}$$

On the other hand a non-Hermitian Hamiltonian H is said to be η-pseudo-Hermitian (thus possessing real or complex conjugate pairs of energies), if [8]

$$H = H^\sharp = \eta^{-1}H^\dagger\eta \tag{5}$$

where η is a linear, Hermitian, invertible operator.

Let a non-Hermitian Hamiltonian $H_1(x;a)$

$$H_1(x;a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x;a) \tag{6}$$

be defined in such a way that the potential $V_1(x;a)$ has an even real part $V_+(x;a)$ and an odd imaginary part $V_-(x;a)$:

$$V_1(x;a) = V_+(x;a) + \mathrm{i}V_-(x;a), \qquad V_\pm(\pm x) = \pm V_\pm(x). \tag{7}$$

Evidently, $H_1(x;a)$ is $\mathcal{PT}$ symmetric,

$$\mathcal{PT}H_1(x;a) = H_1(x;a)\mathcal{PT} \tag{8}$$

and for such a Hamiltonian, η may be represented by the parity operator $\mathcal{P}$, i.e., $H_1(x;a)$ is $\mathcal{P}$-pseudo-Hermitian.

Now the Hamiltonian in (1) can always be factorized using the following ansatz [15]:

$$H_1 = BA + E_0^{(1)} \tag{9}$$

where A and B are defined by

$$\left.\begin{aligned} A &= \frac{\mathrm{d}}{\mathrm{d}x} + W(x;a) \\ B &= -\frac{\mathrm{d}}{\mathrm{d}x} + W(x;a) \end{aligned}\right\} \tag{10}$$

$W(x;a)$ being given in terms of the ground state eigenfunction $\psi_0^{(1)}(x;a)$ of H_1:

$$W(x;a) = -\frac{\psi_0^{(1)\prime}(x;a)}{\psi_0^{(1)}(x;a)}. \tag{11}$$

This allows H_1 to be identified with the well-known form

$$H_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2 - W' + E_0^{(1)} \tag{12}$$

where $E_0^{(1)}$ is the ground state energy of H_1.

One can then construct another Hamiltonian H_2, isospectral to H_1, by

$$H_2 = AB + E_0^{(1)} \tag{13}$$

which, in terms of $W(x;a)$, reduces to

$$H_2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2 + W' + E_0^{(1)}. \tag{14}$$

Evidently, if $\psi_n^{(1)}$ is an eigenfunction of H_1 with energy eigenvalue $E_n^{(1)}$, then $\psi_n^{(2)} = A\psi_n^{(1)}$ is an eigenfunction of H_2 with the same eigenvalue $E_n^{(1)}$, except for the ground state, which is annihilated by A.

$$H_2 A\psi_n^{(1)} = (AB)A\psi_n^{(1)} = A(BA)\psi_n^{(1)} = A\big(H_1\psi_n^{(1)}\big) = E_n^{(1)}\big(A\psi_n^{(1)}\big). \tag{15}$$

Thus,

$$E_{n+1}^{(1)} = E_n^{(2)}, \qquad \psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)} - E_0^{(1)}}} A\psi_{n+1}^{(1)}. \tag{16}$$

Thus A and B play the role of intertwining operators for the partner Hamiltonians H_1 and H_2:

$$AH_1 = H_2 A, \qquad H_1 B = BH_2 \tag{17}$$

$A(B)$ converts an eigenfunction of H_1 (H_2) into an eigenfunction of H_2 (H_1), with the same energy. Additionally, $A(B)$ destroys (creates) an extra node in the eigenfunction.

For conventional Hermitian quantum systems, $W(x;a)$ is the superpotential and $B = A^\dagger$. However, for non-Hermitian systems in general, $B \neq A^\dagger$, as $W(x;a)$ is a complex function. In analogy with conventional quantum mechanics, and considering the η-pseudo-Hermiticity of the Hamiltonian, $W(x;a)$ may be termed as the *pseudo-superpotential*.

Let us now construct a matrix Hamiltonian $\mathcal{H}$, of the form

$$\mathcal{H} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix}. \tag{18}$$

If we consider the following matrix representation for η [8]

$$\eta = \begin{pmatrix} \eta_+ & 0 \\ 0 & \eta_- \end{pmatrix} \tag{19}$$

where $\eta_+(\eta_-)$ is a Hermitian linear automorphism of $H_2(H_1)$, it follows from (5), that the intertwining operators A and B must be related through

$$B = A^\sharp = \eta_+^{-1} A^\dagger \eta_- \tag{20}$$

Hence, the pseudo-superpotential $W(x;a)$ must obey the relationship

$$W(x;a) = \eta_+^{-1} W^*(x;a)\eta_- \tag{21}$$

which, for the $\mathcal{PT}$ symmetric Hamiltonian $H_1(x;a)$ considered here (with $\eta_\pm = \pm\mathcal{P}$), reduces to

$$(\mathcal{PT})W(x;a)(\mathcal{PT})^{-1} = -W(x;a). \tag{22}$$

Writing $W(x; a)$ in the form

$$W(x; a) = W_R(x; a) + \mathrm{i}W_I(x; a) \tag{23}$$

the condition (22) implies

$$\mathcal{P}W_R(x; a)\mathcal{P}^{-1} = -W_R(x; a), \qquad \mathcal{P}W_I(x; a)\mathcal{P}^{-1} = W_I(x; a). \tag{24}$$

Thus the matrix Hamiltonian $\mathcal{H}$ constructed above represents the pseudo-supersymmetric Hamiltonian, formed by the pseudo-supersymmetric partners H_1 and H_2,

$$\mathcal{H} = \begin{pmatrix} H_2 & 0 \\ 0 & H_1 \end{pmatrix} = \begin{pmatrix} AA^\sharp & 0 \\ 0 & A^\sharp A \end{pmatrix}. \tag{25}$$

The pseudo-super-Hamiltonian $\mathcal{H}$ is part of a closed algebra containing both bosonic and fermionic operators, with commutation and anticommutation relations. Such a quantum system is generated by pseudo-supercharges Q and $Q^\sharp$, which change bosonic degrees of freedom into fermionic ones and vice versa:

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\sharp = \begin{pmatrix} 0 & 0 \\ A^\sharp & 0 \end{pmatrix} = \eta^{-1}Q^\dagger\eta. \tag{26}$$

The following commutation and anticommutation relations then describe the closed pseudo-superalgebra

$$\mathcal{H} = \{Q, Q^\sharp\}, \qquad Q^2 = Q^{\sharp 2} = 0, \qquad [Q, \mathcal{H}] = [Q^\sharp, \mathcal{H}] = 0. \tag{27}$$

Let the dependence of the potential $V_1(x; a)$ on the set of parameters a be such that spontaneous breakdown of $\mathcal{PT}$ symmetry occurs at some critical value of a, say a_c, and real energies change to complex conjugate pairs. In terms of the pseudo-superpotential, the condition (22) or (24) holds only for unbroken $\mathcal{PT}$ symmetry. In such a situation, the relationship (20) breaks down: $B \neq A^\sharp$. Consequently, the isospectrality of the partners is lost as A and $A^\sharp$ fail to intertwine the non-Hermitian Hamiltonians denoted by H_2 and H_1. Though one can still write $H_2 = AB$ formally, the anticommutator of the pseudo-supercharges fails to give the pseudo-super-Hamiltonian $\mathcal{H}$

$$\{Q, Q^\sharp\} \neq \mathcal{H}. \tag{28}$$

In analogy with the spontaneous breakdown of supersymmetry in conventional quantum mechanics (with vanishing zero energy ground state), this may be viewed as the explicit breakdown of pseudo-supersymmetry in non-Hermitian $\mathcal{PT}$ symmetric quantum systems. Thus the pseudo-supersymmetric algebra defined in (27) holds only for unbroken $\mathcal{PT}$ symmetry, when the pseudo-superpotential defined in (11) above obeys (24), and the energies are real. However, at the point of spontaneous breakdown of $\mathcal{PT}$ symmetry ($a = a_c$), when the energies of the system switch from real to complex conjugate pairs, both conditions (20) and (24) are violated, and the pseudo-supersymmetry of the system is explicitly broken.

We consolidate our observations with a couple of exactly solvable examples.

3. Explicit examples

3.1. $\mathcal{PT}$ symmetric Scarf II potential

The non-Hermitian $\mathcal{PT}$ symmetric Scarf II model may be described by the Hamiltonian

$$H_1(x; v_1, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - v_1\,\mathrm{sech}^2 x - \mathrm{i}\left(v_1 + a + \frac{1}{4}\right)\mathrm{sech}\,x\tanh x, \qquad v_1 > 0 \tag{29}$$

where v_1 and a are real. The energy levels and the corresponding eigenfunctions are given by [6]

$$E_{nq}^{(1)}(v_1;a) = -\left\{n+\frac{1}{2}-\frac{1}{2}(s+qt)\right\}^2, \qquad n=0,1,2,\ldots<\frac{1}{2}(|s+qt|-1) \tag{30}$$

$$\psi_{nq}^{(1)}(x;v_1,a) = N_{nq}\left(\frac{1-\mathrm{i}\sinh x}{2}\right)^{-\lambda_q}\left(\frac{1+\mathrm{i}\sinh x}{2}\right)^{-\mu_q} P_n^{-2\lambda_q-\frac{1}{2},-2\mu_q-\frac{1}{2}}(\mathrm{i}\sinh x) \tag{31}$$

where $s=\sqrt{2v_1+a+\frac{1}{2}}$, $t=\sqrt{-a}$, $\lambda_q=-\frac{1}{4}+q\frac{s}{2}$, $\mu_q=-\frac{1}{4}+q\frac{t}{2}$ and q $(=\pm1)$ is the quasiparity, giving rise to doublet solutions, which is a characteristic feature of this class of $\mathcal{PT}$ symmetric models. Normalization requirement restricts the signs allowed in λ_q and μ_q.

It follows from (29) and (31) that the Hamiltonian $H_1(x;v_1,a)$ is always invariant under the $\mathcal{PT}$ transformation irrespective of the value of a, while the wavefunctions $\psi_{nq}^{(1)}(x;v_1,a)$ are $\mathcal{PT}$ invariant only when

$$-\left(2v_1+\tfrac{1}{2}\right)\leqslant a\leqslant 0. \tag{32}$$

The pseudo-superpotential corresponding to the Hamiltonian in (29) above, may be given by

$$\begin{aligned}W(x;a) &= (\lambda_q+\mu_q)\tanh x-\mathrm{i}(\lambda_q-\mu_q)\operatorname{sech} x\\ &= \frac{1}{2}\left(-1+\sqrt{2v_1+a+\frac{1}{2}}+q\sqrt{-a}\right)\tanh x-\frac{\mathrm{i}}{2}\left(\sqrt{2v_1+a+\frac{1}{2}}-q\sqrt{-a}\right)\operatorname{sech} x.\end{aligned} \tag{33}$$

Obviously, (24) is satisfied for real λ_q and μ_q, which, in turn, is related to (32), and hence to unbroken $\mathcal{PT}$ symmetry, i.e. real energies. At the same time whenever a crosses a critical value a_c, i.e., a lies beyond the region specified in (32), and energies switch to complex conjugate pairs, two simultaneous phenomena are observed:

(i) the condition (24) is violated, thus inducing spontaneous breakdown of $\mathcal{PT}$ symmetry in $H_1(x;v_1,a)$;

(ii) the violation of (20) leading to the explicit breakdown of pseudo-supersymmetry.

If one keeps v_1 fixed, then from (30) one can show that though

$$\lim_{a\to0^-}E_{nq}^{(1)}(a)=E_{nq}^{(1)}(a=0) \tag{34}$$

the right-hand limit, viz., $\lim_{a\to0^+}E_{nq}^{(1)}(a)$, does not exist. A similar situation occurs at $a=-(2v_1+1/2)$.

It would be interesting to study the nature and behaviour of the partner Hamiltonian $H_2(x;v_1,a)$, from (14).

(i) For a lying in the range as given in (32),

$$\begin{aligned}H_2(x;v_1,a) &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\left\{-\frac{3}{4}+\frac{s^2+t^2}{2}-(s+qt)\right\}\operatorname{sech}^2 x-\mathrm{i}\\ &\quad\times\left\{\frac{1}{2}(s^2-t^2)-(s-qt)\right\}\operatorname{sech} x\tanh x.\end{aligned} \tag{35}$$

Evidently, as the condition (24) is obeyed in this case, the partner Hamiltonian $H_2(x;v_1,a)$ is also $\mathcal{PT}$ symmetric. It has real energies, isospectral to $H_1(x;v_1,a)$, with the possible exception of the ground state. Thus $H_1(x;v_1,a)$ and $H_2(x;v_1,a)$ form the pseudo-supersymmetric partners of the super-Hamiltonian $\mathcal{H}$, obeying the pseudo-supersymmetric algebra given in (27).

(ii) For values of a outside the range given in (32), $\mathcal{PT}$ symmetry is spontaneously broken in the Scarf II Hamiltonian $H_1(x;v_1,a)$. Let $a>0$, so that $t=\mathrm{i}\alpha$. It can be seen that the partner Hamiltonian $H_2(x; v_1, a)$ is no longer $\mathcal{PT}$ symmetric:

$$H_2(x; v_1, a) = -\frac{d^2}{dx^2} - \left\{-\frac{3}{4} + \frac{s^2 - \alpha^2 - 2s}{2} - iq\alpha\right\} \operatorname{sech}^2 x - i \times \left\{\frac{1}{2}(s^2 + \alpha^2 - 2s) + iq\alpha\right\} \operatorname{sech} x \tanh x. \quad (36)$$

Thus the spontaneous breakdown of $\mathcal{PT}$ symmetry in the Scarf II Hamiltonian $H_1(x; v_1, a)$ is manifested as explicit $\mathcal{PT}$ symmetry breaking in the partner Hamiltonian $H_2(x; v_1, a)$, the two no longer being isospectral to each other. Though one can still write $H_2 = AB$ formally, the pseudo-supersymmetry is explicitly broken. Thus the spontaneous breakdown of $\mathcal{PT}$ symmetry is accompanied by the explicit breakdown of pseudo-supersymmetry.

The wavefunctions, too, behave quite strangely at these points of spectral discontinuities. So long as $\mathcal{PT}$ symmetry is unbroken, the wavefunctions are normalizable in the sense of $\mathcal{CPT}$ norm [11, 16]:

$$\langle\psi_m|\psi_n\rangle^{\mathcal{CPT}} = \int dx\, \psi_m^{\mathcal{CPT}}(x)\psi_n(x) = \delta_{m,n}, \qquad \psi_m^{\mathcal{CPT}}(x) = \int dy\, \mathcal{C}(x, y)\psi_m^*(y) \quad (37)$$

where $\mathcal{C}$ is the charge operator. The interesting point to be observed here is that, at the point of spontaneous breakdown of $\mathcal{PT}$ symmetry, though the wavefunctions remain well behaved, their $\mathcal{CPT}$ norm vanishes:

$$\int (\mathcal{CPT}\psi_n(x))\psi_n(x)\, dx \to 0. \quad (38)$$

This can be shown by straightforward calculations [17]. Thus, unlike the Hermitian models [9] where the effect of spectral discontinuities forces the eigenfunction to be non-square integrable, in the present case the eigenfunctions, though exhibiting proper behaviour at $\pm\infty$, become *self-orthogonal* [3].

3.2. $\mathcal{PT}$ symmetric oscillator

We next consider another non-Hermitian model, $\mathcal{PT}$ symmetrized in a different way; viz., the well-known $\mathcal{PT}$ symmetric oscillator, given by the Hamiltonian

$$H_1(x; a) = -\frac{d^2}{dx^2} + (x - i\epsilon)^2 + \frac{a - \frac{1}{4}}{(x - i\epsilon)^2} \quad (39)$$

where ϵ is a real number. The energy eigenvalues and the corresponding eigenfunctions are given by [18]

$$E_{nq}^{(1)}(a) = 4n + 2 - 2q\sqrt{a} \qquad n = 0, 1, 2, \ldots \quad (40)$$

$$\psi_{nq}(x; a) = N_{nq}\, e^{-\frac{(x-i\epsilon)^2}{2}} (x - i\epsilon)^{-q\sqrt{a}+\frac{1}{2}} L_n^{(-q\sqrt{a})}((x - i\epsilon)^2) \quad (41)$$

where the quasiparity $q(= \pm 1)$ again gives doublet solutions. Proceeding in a similar manner, the pseudo-superpotential, $W(x; a)$, and the partner, $H_2(x; a)$, turn out to be

$$W(x; a) = (x - i\epsilon) - \frac{-q\sqrt{a} + \frac{1}{2}}{(x - i\epsilon)} \quad (42)$$

$$H_2(x; a) = -\frac{d^2}{dx^2} + (x - i\epsilon)^2 + \frac{a - 2q\sqrt{a} + \frac{3}{4}}{(x - i\epsilon)^2} + 2. \quad (43)$$

Thus, it is easy to observe that the critical value of a here is $a_c = 0$. So long as

$$a \geqslant 0 \tag{44}$$

the condition (24) is satisfied, $\mathcal{PT}$ symmetry is unbroken in the $\mathcal{PT}$ oscillator, and the partner $H_2(x; a)$ Hamiltonian (in (43)) is also $\mathcal{PT}$ symmetric, both sharing same real energies, without possibly the ground state. Consequently, pseudo-supersymmetry is unbroken. On the other hand, for $a < 0$, $\mathcal{PT}$ symmetry is spontaneously broken in the original Hamiltonian, giving complex conjugate energies. The conditions (20) and (24) are violated, leading to the explicit breakdown of pseudo-supersymmetry. Furthermore, though

$$\lim_{a\to 0^+} E_{nq}^{(1)}(a) = E_{nq}^{(1)}(0) \tag{45}$$

the left-hand limit, viz., $\lim_{a\to 0^-} E_{nq}^{(1)}(a)$, does not exist. Additionally, though the wavefunctions remain well behaved at $\pm\infty$, their $\mathcal{CPT}$ norm goes to zero. Thus in this model too, the point of discontinuity of the spectrum is associated with the simultaneous breakdown of $\mathcal{PT}$ symmetry and pseudo-supersymmetry.

4. Conclusions

In the present letter we have established the relation between the spontaneous breakdown of $\mathcal{PT}$ symmetry and the explicit breakdown of pseudo-supersymmetry, at some critical value a_c of a set of parameters a in the Hamiltonian $H(x; a)$. In particular, we have shown that in a class of non-Hermitian, but $\mathcal{PT}$ symmetric Hamiltonians $H_1(x; a)$, the changing of energies from real to complex conjugate values is a direct consequence of the simultaneous breakdown of these two symmetries. The anticommutator of the pseudo-supercharges Q and $Q^\sharp$ fails to give the pseudo-super-Hamiltonian $\mathcal{H}$, as the Hamiltonian $H_2 = AA^\sharp$ is no longer isospectral to its partner $H_1 = A^\sharp A$. In fact, $\mathcal{PT}$ symmetry is explicitly broken in the partner $H_2(x; a)$. Furthermore, though the wavefunctions remain well behaved, they become self-orthogonal beyond a_c, as their $\mathcal{CPT}$ norm goes to zero. All the above observations hold in both the explicit examples considered here.

Acknowledgments

This work was partly supported by SERC, DST, Government of India, through the Fast Track Scheme for Young Scientists (DO No. SR/FTP/PS-07/2004), to one of the authors (AS).

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M and Boettcher S 1998 *J. Phys. A: Math. Gen.* **31** L273
[2] Hatano N and Nelson D R 1996 *Phys. Rev.* B **58** 8384
Heiss W D 2002 *Preprint* quant-ph/0211090
Heiss W D 2003 *Preprint* quant-ph/0304152
[3] Narevicius E, Serra P and Moiseyev N 2003 *Eur. Phys. Lett.* **62** 789
[4] ’t Hooft G and Nobbenhuis S 2006 *Preprint* gr-qc/0602076
[5] Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 4561
Lévai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165
Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** 5679
Bender C M, Boettcher S, Jones H F, Meisinger P N and Simsek M 2001 *Phys. Lett.* A **291** 197
[6] Ahmed Z 2001 *Phys. Lett.* A **282** 343
Ahmed Z 2001 *Phys. Lett.* A **287** 295

[7] Bagchi B and Quesne C 2000 *Phys. Lett.* A **273** 285
Bagchi B and Quesne C 2002 *Phys. Lett.* A **300** 18
[8] Mostafazadeh A 2002 *Nucl. Phys.* B **640** 419
Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2002 *J. Math. Phys.* **43** 3944
Mostafazadeh A 2003 *J. Math. Phys.* **44** 974
[9] Herbst I W and Simon B 1978 *Phys. Lett.* B **78** 304
Calogero F 1979 *Lett. Nuovo. Cimento* **25** 533
Saxena R P and Varma V S 1982 *J. Phys. A: Math. Gen.* **15** L149
Saxena R P, Srivastava P K and Varma V S 1988 *J. Phys. A: Math. Gen.* **21** L389
Pandey R K and Varma V S 1989 *J. Phys. A: Math. Gen.* **22** 459
[10] Turbiner A 1991 *Phys. Lett.* B **276** 95
[11] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401
Bender C M, Brody D C and Jones H F 2004 *Phys. Rev. Lett.* **92** 119902
Mondal C K, Maji K and Bhattacharyya S P 2001 *Phys. Lett.* A **291** 203
Bender C M and Monou M 2005 *J. Phys. A: Math. Gen.* **38** 2179
[12] Levai G and Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 8793
[13] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** L391
[14] Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 *Phys. Lett.* B **483** 284
Levai G and Znojil M 2001 *Mod. Phys. Lett.* A **16** 1973
Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 2341
[15] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[16] Bender C M, Meisinger P N and Wang Q 2003 *J. Phys. A: Math. Gen.* **36** 1973
Bender C M, Brod J, Refig A and Reuter M E 2004 *J. Phys. A: Math. Gen.* **37** 10139
[17] Levai G, Cannata F and Ventura A 2002 *Phys. Lett.* A **300** 271
[18] Znojil M 1999 *Phys. Lett.* A **259** 220